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Exercises for 'Functional Analysis 2' [MATH-404]

(19/05/2025)

Ex 12.1 (An algorithm for Edelstein's fixed point theorem)
We saw in the lecture that when (M, d) is a compact metric space and $F : M \to M$ satisfies $d(F(x), F(y)) < d(x, y)$ for all $x \neq y$, then F has a unique fixed point $\bar{x} \in M$. Show that for any $x_0 \in M$ the iteratively defined sequence $x_{n+1} = F(x_n)$ converges to $\bar{x}$.

Solution 12.1 : It suffices to show that every subsequence contains a subsequence that converges to the fixed point. By compactness, assume that there exists a subsequence x_{n_k} that converges to a point $z \neq \bar{x}$. Consider the real-valued sequence $y_n = d(x_n, x_{n+1})$. Then

$$y_{n+1} = d(x_{n+1}, x_{n+2}) = d(F(x_n), F(x_{n+1})) \leq d(x_n, x_{n+1}) = y_n.$$

Hence $(y_n)_{n\in\mathbb{N}}$ is monotone decreasing and non-negative and we conclude that there exists a limit $\bar{y} = \lim_{n\to+\infty} y_n$. Hence

$$\begin{aligned}\bar{y} &= \lim_{k\to+\infty} y_{n_k+1} = \lim_{k\to+\infty} d(x_{n_k+1}, F(x_{n_k+1})) = \lim_{k\to+\infty} d(F(x_{n_k}), F(F(x_{n_k})))\\ &= d(F(z), F(F(z))) < d(z, F(z)) = \lim_{k\to+\infty} d(x_{n_k}, F(x_{n_k})) = \lim_{k\to+\infty} y_{n_k} = \bar{y},\end{aligned}$$

which yields a contradiction.

Ex 12.2 (Schaefer's fixed point theorem)
Let X be a Banach space and $F : X \to X$ be continuous such that $\overline{F(B)}$ is compact for every bounded set $B \subset X$. Assume further that there exists $R > 0$ such that

$$\{x \in X : x = \lambda F(x) \text{ for some } \lambda \in [0, 1]\} \subset B_R(0).$$

Show that F has a fixed point.
Hint: Define the projection operator $p_R : X \to \overline{B_R(0)}$ by $p_R(x) = x$ on $\overline{B_R(0)}$ and $p_R(x) = R\frac{x}{|x|}$ otherwise and consider the map $F_R = p_R \circ F$. Apply Schauder's fixed point theorem on a suitable set.

Solution 12.2 : As suggested in the hint, we consider $F_R : \overline{B_R(0)} \to \overline{B_R(0)}$ defined by $F_R(x) = p_R(F(x))$. The map F_R is continuous since p_R and F are continuous. Note that $\overline{F_R(\overline{B_R(0)})}$ is compact since the image of a precompact set under a continuous function is precompact.[1] Moreover, $\overline{B_R(0)}$ is closed and convex. By Schauder's fixed point theorem the

1. Indeed, if $A \subset X$ is precompact and $f : X \to Y$ is continuous, then $f(\overline{A})$ is compact, hence closed (this is true even if Y is just a Hausdorff space). Hence $\overline{f(A)} \subset f(\overline{A})$. Therefore $f(A)$ is precompact since its closure is a closed subset of a compact subset.

map F_R has a fixed point $x_0 \in \overline{B_R(0)}$. We claim that x_0 is also a fixed point of F. This is immediate if $\|F(x_0)\| \leq R$. Assume by contradiction that $\|F(x_0)\| > R$. Then

$$x_0 = p_R(F(x_0)) = R\frac{F(x_0)}{\|F(x_0)\|} = \underbrace{\frac{R}{\|F(x_0)\|}}_{\in[0,1]} F(x_0) \in B_R(0) \cap \partial B_R(0),$$

which gives a contradiction.

Ex 12.3 (Peano's existence theorem for ODEs*)
Let $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ and consider the Cauchy problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \tag{1}$$

Assume that $f : [-a + t_0, a + t_0] \times \overline{B_R(y_0)} \to \mathbb{R}^n$ is continuous. Show that there exists $\delta > 0$ such that the Cauchy problem (1) has a solution $y : [-\delta + t_0, \delta + t_0] \to \mathbb{R}^n$.
Hint: Apply the Schauder fixed point theorem to the integral operator $y \mapsto y_0 + \int_{t_0}^t f(s, y(s))\,\mathrm{d}s$. Use the Arzelà–Ascoli theorem to show the compactness of the operator.

Ex 12.4 (Existence of solutions for a periodic BVP)
Let $\mu \in \mathbb{R} \setminus \{0\}$ and $\mathbb{J} = [0, T]$ for some $T > 0$.

a) Consider the linear first order periodic boundary value problem

$$u'(t) + \mu u(t) = f(t), \quad t \in J, \quad u(0) = u(T),$$

where $f \in C(\mathbb{J})$. Find the Green's function $g(t, s)$ such that

$$u(t) = [Gf](t) := \int_0^T g(t, s) f(s)\,\mathrm{d}s, \quad t \in \mathbb{J},$$

is a solution to this problem.
Hint: Consider the function $y(t) = e^{\mu t}u(t)$.

b) Show that $G\colon C(\mathbb{J}) \to C(\mathbb{J})$ is continuous and maps bounded subsets of $C(\mathbb{J})$ into relatively compact sets.

c) Assume that $f\colon \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is continuous and has *sublinear growth*, i.e.

$$|f(t, u)| \leq a(t) + b|u|^\alpha,$$

where $a \in C(\mathbb{J})$, $b > 0$, and $\alpha \in [0, 1)$. Applying Schaefer's theorem show that there exists a solution to the following nonlinear first order periodic problem

$$u'(t) + \mu u(t) = f(t, u(t)), \quad t \in \mathbb{J}, \quad u(0) = u(T).$$

Solution 12.4 :
a) Assume that u solves the linear equation and let $y(t) = e^{\mu t}u(t)$, $t \in \mathbb{J}$. Then y satisfies $y'(t) = e^{\mu t}f(t)$ for $t \in \mathbb{J}$, with $y(T) = e^{\mu T}y(0)$. Thus, integrating the differential identity

$$y(t) = y(0) + \int_0^t e^{\mu s} f(s)\mathrm{d}s$$

and using this expression for $t = T$ together with the boundary relation we get

$$y(0) = \frac{1}{e^{\mu T} - 1} \int_0^T e^{\mu s} f(s)\mathrm{d}s.$$

In consequence

$$u(t) = e^{-\mu t} y(t) = \frac{1}{1 - e^{-\mu T}} \Big[\int_t^T e^{-\mu(T+t-s)} f(s)\,\mathrm{d}s + \int_0^t e^{-\mu(t-s)} f(s)\,\mathrm{d}s \Big],$$

which can be succinctly written as

$$u(t) = \int_0^T g(t,s) f(s)\mathrm{d}s, \; t \in \mathbb{J} \quad \text{where } g(t,s) = \frac{1}{1 - e^{-\mu T}} \begin{cases} e^{-\mu(t-s)}, & 0 \le s \le t \le T, \\ e^{-\mu(T+t-s)}, & 0 \le t < s \le T. \end{cases}$$

b) To analyze the operator G, one can use the upper formula with two integrals, which have dependence on t also in the integration limits, or the lower representation via the Green's function $g(t,s)$, which has a jump discontinuity on the diagonal $s = t$ of size $e^{-\mu T}$. Here, we proceed with the latter to illustrate that the integral operators with bounded discontinuous kernels (at least on a null set in $\mathbb{R}^2$) can be still well-defined and compact.

Because g is bounded, the sup-norm $\|g\|_\infty$ on $\mathbb{J} \times \mathbb{J}$ is finite. Fix $t_0 \in \mathbb{J}$ and for $\varepsilon > 0$ let us denote $\mathbb{D}_\varepsilon = \{s \in \mathbb{J} : |t_0 - s| < \varepsilon\}$. Then if $v \in C(\mathbb{J})$, for any $t \in \mathbb{J}$ we can write

$$\begin{aligned} \big|[Gv](t_0) - [Gv](t)\big| &\le \int_0^T |g(t_0,s) - g(t,s)|\,\mathrm{d}s \cdot \|v\|_{C(\mathbb{J})} \\ &\le \Big(\int_{\mathbb{J}\setminus\mathbb{D}_\varepsilon} |g(t_0,s) - g(t,s)|\,\mathrm{d}s + 2\|g\|_\infty\,\varepsilon \Big) \cdot \|v\|_{C(\mathbb{J})} \end{aligned}$$

Since g is continuous on $\overline{\mathbb{D}}_\varepsilon \times (\mathbb{J} \setminus \mathbb{D}_\varepsilon)$, thus uniformly continuous, and ε can be made arbitrary small, $Gv \in C(\mathbb{J})$. That G is continuous as an operator on $C(\mathbb{J})$ follows from

$$\|Gv\|_{C(\mathbb{J})} \le \|g\|_\infty \|v\|_{C(\mathbb{J})},$$

for any $v \in C(\mathbb{J})$. The last inequality implies also that if $\Phi \subset C(\mathbb{J})$ is bounded, we have

$$\sup\{\|Gv\|_{C(\mathbb{J})} : \; v \in \Phi\} < \infty.$$

Additionally, taking the supremum over Φ in the preceding estimate we arrive at

$$\sup_{v\in\Phi} \big|[Gv](t_0) - [Gv](t)\big| \le \Big(\int_{\mathbb{J}\setminus\mathbb{D}_\varepsilon} |g(t_0,s) - g(t,s)|\,\mathrm{d}s + 2\|g\|_\infty\,\varepsilon \Big) \cdot \sup_{v\in\Phi} \|v\|_{C(\mathbb{J})},$$

and invoking the uniform continuity of g on $\overline{\mathbb{D}}_\varepsilon \times (\mathbb{J} \setminus \mathbb{D}_\varepsilon)$ once more, we see that $G(\Phi)$ is equicontinuous. By the Arzela-Ascoli theorem, $G(\Phi)$ has compact closure.

c) Let us denote

$$u \mapsto N(u) : N(u)(t) = f(t, u(t)), \; t \in \mathbb{J}$$

the nonlinear Nemytski (superposition) operator. Because f is continuous, $N\colon C(\mathbb{J}) \to C(\mathbb{J})$ and N maps bounded sets into bounded ones. To verify the latter statement, take any bounded $\Phi \subset C(\mathbb{J})$ and let $K = \sup_{v\in\Phi} \|v\|_{C(\mathbb{J})}$. Then, f is bounded on $\mathbb{J} \times [-K, K]$, say by L, and so

$$\|N(v)\|_{C(\mathbb{J})} \le L, \quad \text{for all } v \in \Phi.$$

According to a) and b), the nonlinear periodic problem can be equivalently written as a fixed value problem (note that the periodic boundary is encoded in the operator G)

$$u = [G \circ N](u), \quad u \in C(\mathbb{J}).$$

From the preceding, we know that $F = G \circ N$ is continuous on $C(\mathbb{J})$ and maps bounded subsets of $C(\mathbb{J})$ into relatively compact sets. It remains to establish the *a priori* bound on the solutions to

$$u = \lambda F(u), \quad \lambda \in [0,1]. \tag{$\star$}$$

If u is the solution to ($\star$), then for $t \in \mathbb{J}$ we have that

$$|u(t)| \leq \lambda \int_0^T |g(t,s)| \left|[N(u)](s)\right| \mathrm{d}s \leq \|g\|_\infty \int_0^T |a(s)| + b|u(s)|^\alpha \, \mathrm{d}s.$$

Therefore, there are constants $a^*, b^* > 0$ such that

$$\|u\|_{C(\mathbb{J})} \leq a^* + b^* \|u\|_{C(\mathbb{J})}^\alpha.$$

Taking into account that $0 \leq \alpha < 1$, we can conclude that there exist a constant $C > 0$ such that $\|u\|_{C(\mathbb{J})} \leq C$ for any solution to ($\star$) with $\lambda \in [0,1]$. By Schaefer's theorem we obtain the existence of a fixed point for F.